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Second-order dichotomous processes: Damped free motion, critical behavior, and anomalous superdiffusion

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We study damped inertial processes driven by dichotomous Markov noise in the absence of a potential. We obtain exact differential equations for the joint probability density and for the marginal densities of velocity and position. Several aspects of the critical behavior of the system are examined in detail. The exact equation for the displacement of a free Brownian particle is also found, and from this equation we study the damping effects on the mean-square displacement $\langle X^2(t)\rangle$, by evaluating the dynamical exponent and showing a crossover from a superdiffusive motion of the form $\langle X^2(t)\rangle \sim t^3$ to ordinary diffusion where $\langle X^2(t)\rangle \sim t$.

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I. INTRODUCTION

We call second-order processes (also called inertial processes) to random processes $X(t)$ whose dynamical evolution is governed by equations of the form

$$\ddot{X} + \beta\dot{X} + f(X) = F(t), \tag{1.1}$$

where $-f(X)$ is the force that determines the motion of the isolated system and the other terms arise from the interaction of the system with the surroundings as determined by the damping force $\beta\dot{X}$ and a fluctuating force $F(t)$ with known statistical properties. Even in the simplest cases, the processes described by (1.1) are very difficult to deal with mainly due to mathematical complexity. One example of this is provided by the evaluation of the probability density function for the output of second-order filters driven by the random telegraph signal [1] where the only results available are Monte Carlo simulations and other approximate results for second-order Butterworth filters [2]. A large and dispersed literature surrounds this problem when the driving noise $F(t)$ is Gaussian. However, exact solutions and even exact equations are essentially unavailable [3,4]. The usual approach (sometimes called "the adiabatic approximation") consists in assuming that the damping coefficient β is very large, so that the acceleration, $\ddot{X}$, becomes negligible and the evolution equation reduces to that of a first-order process.

In a recent paper [5] we have studied the process (1.1) in the driftless undamped case:

$$\ddot{X} = F(t), \tag{1.2}$$

where $F(t)$ is dichotomous Markov noise alternately taking on values $\pm a$ with an exponential switch probability density function

$$\psi(t) = \lambda e^{-\lambda t}, \tag{1.3}$$

where λ^{-1} is the average time between switches. We note that this $F(t)$ is a colored noise with the correlation function given by

$$\langle F(t)\,F(t')\rangle = a^2\, e^{-|t-t'|/\tau_c}, \tag{1.4}$$

where

$$\tau_c \equiv \frac{1}{2\lambda} \tag{1.5}$$

is the correlation time.

Our focus in [5] has been on the joint density $p(x,y,t)$ for the probability that the position $X(t)$ lies between x and $x+dx$ and that the velocity $\dot{X}(t)$ lies between y and $y+dy$. We have also obtained exact equations for the marginal probability density $p(y,t)$ of the velocity and the marginal probability density $p(x,t)$ of the position. The equation satisfied by $p(y,t)$ is the ordinary telegrapher's equation with constant coefficients while the equation satisfied by $p(x,t)$ is a telegrapher's equation with variable coefficients. In the Gaussian white-noise limit, this latter equation reduces to a diffusion equation with a time-dependent diffusion coefficient which leads to anomalous "superdiffusive" motion of the form

$$\langle X^2(t)\rangle \sim t^3$$

to be contrasted with ordinary diffusion where the exponent of time is 1 and with deterministic motion where the exponent is 2. The results of this analysis have been recently applied [6] to the problem of chemical reactions in constrained geometries where, in some cases, the kinetics of the reaction is "anomalous" in low dimensions in the sense of being different from the results of mass action [7]. Most of the analysis of this problem deals with the diffusion-limited regime, in which the particles diffuse through the system and react instantaneously upon contact. If we think of the motion of the particles as being governed by a second-order process such as (1.1)

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but with no drift term $f(X)$ then the diffusion-limited regime is the high-damping limit, that is, the adiabatic approximation. The contribution of Araujo *et al.* [6] is the investigation of these systems in the low-damping limit, that is, as described by Eq. (1.2). They find that the reaction kinetics changes drastically from that of the high-damping regime, thus introducing a new series of "anomalies" that had previously not been noticed.

It is precisely to consider the effect of the damping on its complete scale that we will now extend our previous results for the undamped case (1.2) to the more general inertial process:

$$\ddot{X} + \beta \dot{X} = F(t), \tag{1.6}$$

where $F(t)$ is dichotomous Markov noise. Herein we obtain exact differential equations and, in some cases, exact analytical solutions for the joint density $p(x,y,t)$ and the marginal densities $p(y,t)$ and $p(x,t)$ of the process described by Eq. (1.6). We also study the asymptotic behavior of these densities and show that $p(y,t)$ and $p(x,t)$ behave in a very different way according to whether the correlation time $(2\lambda)^{-1}$ is greater or smaller than the relaxation time β^{-1}.

We note that when $F(t)$ is Gaussian white noise the process (1.6) represents the free Brownian motion. From the present formalism we also obtain the equation satisfied by the probability density function $p(x,t)$ for the displacement of a free Brownian particle. From this equation we study the damping effects on the mean-square displacement of the system by evaluating the dynamical exponent and showing the crossover from superdiffusion to ordinary diffusion.

The paper is organized as follows. In Sec. II we detail the dynamics of the system and set the general equations. In Sec. III we study the joint density and obtain exact differential equations for this density. Section IV is devoted to the marginal density of the velocity; we obtain its stationary distribution and the exact expression for the characteristic function. In Sec. V the exact equation for the marginal density of the position is presented along with a complete asymptotic analysis. In Sec. VI the free Brownian motion is revisited and we study the regions of superdiffusion and ordinary diffusion. Conclusions are drawn in Sec. VII and more technical aspects of the paper are in the Appendixes.

II. ANALYSIS

From a dynamical point of view the time evolution of the process (1.6) with a dichotomous driving force is given by two different dynamics $x^{\pm}(t;x_0,y_0)$, where $x^{+}(t;x_0,y_0)$ $[x^{-}(t;x_0,y_0)]$ is the solution of Eq. (1.6) with $F(t) = +a$ $[-a]$, and $x_0 = X(0)$ and $y_0 = \dot{X}(0)$ are the initial position and velocity of the process. The system randomly switches between these two dynamics at random times which are governed by a given probability density function $\psi(t)$. The explicit expression for $x^{\pm}(t;x_0,y_0)$ is

$$x^{\pm}(t;x_0,y_0) = x_0 + \frac{y_0}{\beta} \pm \frac{a}{\beta}\left(t - \frac{1}{\beta}\right) - \frac{1}{\beta}\left(y_0 \mp \frac{a}{\beta}\right) e^{-\beta t}. \tag{2.1}$$

Let $Y(t) \equiv \dot{X}(t)$ be the random process representing the velocity. We see from Eq. (1.6) that $Y(t)$ obeys the stochastic differential equation

$$\dot{Y} = -\beta Y + F(t) \tag{2.2}$$

and the dynamics of this first-order process is given by

$$y^{\pm}(t;y_0) = \left(y_0 \mp \frac{a}{\beta}\right) e^{-\beta t} \pm \frac{a}{\beta}. \tag{2.3}$$

Note that $y^{\pm}(t;y_0)$ have asymptotically fixed stable points at $\pm a/\beta$. Hence $y^{+}(t;y_0)$ $[y^{-}(t;y_0)]$ relaxes towards a/β $[-a/\beta]$ with a relaxation time $\tau_r = 1/\beta$. Moreover, if the system does not initially lie in the interval $(-a/\beta, a/\beta)$ then at some later time the system will enter the interval $(-a/\beta, a/\beta)$ with certainty, but once the system is inside the interval it stays there forever. Therefore at sufficiently long times the velocity of our second-order process (1.6) is always bounded by $\pm a/\beta$. There is no such behavior for the position $X(t)$ because, as we see from Eq. (2.1), $x^{\pm}(t;x_0,y_0)$ is not bounded. This is the reason why the velocity $Y(t)$ possesses a stationary distribution while the position $X(t)$ does not.

We will now set the general formalism in order to evaluate the joint density $p(x,y;t)$ of the process (1.6). Two intermediate functions denoted by $\Omega^{+}(x,y;t)$ and $\Omega^{-}(x,y;t)$ will be required in our analysis. They are defined as follows:

$$\Omega^{\pm}(x,y;t)dxdydt = \Pr\{\text{a sojourn with } F(t) = +a(-a) \textit{ ends} \text{ with the process } (X(t),Y(t)) \text{ in the volume } dxdy \text{ during the time interval } (t,t+dt)\}. \tag{2.4}$$

We observe that these functions describe the state of the process at switching times. As we have mentioned, the evolution of the process between switches is deterministic and is given by Eq. (2.1). Therefore the functions $\Omega^{\pm}(x,y;t)$ obey the following set of coupled renewal equations:

$$\Omega^{+}(x,y;t) = \tfrac{1}{2}h^{+}(x,y;t/x_0,y_0) + \int_0^t d\tau \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h^{+}(x,y;\tau/u,v)\Omega^{-}(u,v;t-\tau), \tag{2.5}$$

$$\Omega^{-}(x,y;t) = \tfrac{1}{2}h^{-}(x,y;t/x_0,y_0) + \int_0^t d\tau \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\; h^{-}(x,y;\tau/u,v)\Omega^{+}(u,v;t-\tau). \tag{2.6}$$

The kernels $h^{\pm}(x,y;\tau/u,v)$ appearing on the right-hand sides of these equations are defined by

$$h^{\pm}(x,y;\tau/u,v) \equiv \delta(x-x^{\pm}(\tau;u,v))\delta(y-y^{\pm}(\tau;v))\psi(\tau), \tag{2.7}$$

where $x^{\pm}(\tau;u,v)$ and $y^{\pm}(\tau;v)$ are given as in Eqs. (2.1) and (2.3) and with a similar definition for $h^{\pm}(x,y;t/x_0,y_0)$.

Equations (2.5) and (2.6) are derived from the consideration that when a sojourn in an ocurrence of the plus (minus) state ends at time t, it is either the end of the very first sojourn (accounting for the factor $\frac{1}{2}$ in the first term) or else a sojourn in the minus (plus) state ended at time $t-\tau<t$, at which time the process was in the point (u,v) of the phase space, and the subsequent sojourn in the plus (minus) state lasted for a time τ.

We decompose the joint probability density into two components

$$p(x,y;t) = p^{+}(x,y;t) + p^{-}(x,y;t), \tag{2.8}$$

where, for example, $p^{+}(x,y;t)$ is the probability density for $(X(t),Y(t))$ to be equal to (x,y) at time t while in the plus state, with an analogous definition for $p^{-}(x,y;t)$.

It is not difficult to convince oneself that the densities $p^{\pm}(x,y;t)$ obey similar equations to that of $\Omega^{\pm}(x,y;t)$ just by replacing the ψ's by Ψ's, where $\Psi(t)$ is the probability that the time between switches is greater than t, i.e.,

$$\Psi(t) = \int_t^{\infty} dt'\psi(t'). \tag{2.9}$$

That is,

$$\begin{aligned} p^{\pm}(x,y;t) = {} & \tfrac{1}{2}H^{\pm}(x,y;t/x_0,y_0) \\ & + \int_0^t d\tau \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv\ H^{\pm}(x,y;\tau/u,v) \\ & \times\Omega^{\mp}(u,v;t-\tau), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} H^{\pm}(x,y;\tau/u,v) \equiv {} & \delta(x-x^{\pm}(\tau;u,v)) \\ & \times\delta(y-y^{\pm}(\tau;v))\ \Psi(\tau) \end{aligned} \tag{2.11}$$

and a similar definition for $H^{\pm}(x,y;t/x_0,y_0)$.

Equations (2.5), (2.6), and (2.10) furnish a formal solution to the problem and can be a convenient starting point for numerical analysis when no further analytical treatment can be made.

We finally note that for an exponential switch density, $\psi=\lambda e^{-\lambda t}$, we have

$$\Psi(t) = e^{-\lambda t} \tag{2.12}$$

and

$$p(x,y;t) = \frac{1}{\lambda}\left[\,\Omega^{+}(x,y;t)+\Omega^{-}(x,y;t)\,\right]. \tag{2.13}$$

III. THE JOINT PROBABILITY DENSITY

In what follows we will assume that the switch density has the exponential form given by Eq. (1.3). With this assumption we show in Appendix A that the set of coupled renewal equations (2.5) and (2.6) is equivalent to

$$\frac{\partial}{\partial y}\left[(a-\beta y)\Omega^{+}\right] - y\frac{\partial\Omega^{+}}{\partial x} + \lambda(\Omega^{+}-\Omega^{-}) + \frac{\partial\Omega^{+}}{\partial t} = 0, \tag{3.1}$$

$$\frac{\partial}{\partial y}\left[(a+\beta y)\Omega^{-}\right] + y\frac{\partial\Omega^{-}}{\partial x} + \lambda(\Omega^{+}-\Omega^{-}) - \frac{\partial\Omega^{-}}{\partial t} = 0. \tag{3.2}$$

It also follows from Eqs. (2.5) and (2.6) that the functions $\Omega^{\pm}(x,y;t)$ satisfy the initial conditions

$$\Omega^{\pm}(x,y;0) = \tfrac{1}{2}\lambda\delta(x-x_0)\delta(y-y_0). \tag{3.3}$$

As we have shown in Sec. II, when $\psi(t)$ is exponential the joint density is given by Eq. (2.13). Thus, if we define the auxiliary function

$$q(x,y;t) = \frac{1}{\lambda}\left[\,\Omega^{+}(x,y;t)-\Omega^{-}(x,y;t)\,\right], \tag{3.4}$$

then from Eqs. (3.1) and (3.2) we obtain the system

$$a\frac{\partial p}{\partial y} - \beta\frac{\partial}{\partial y}(y\ q) + y\frac{\partial q}{\partial x} + 2\lambda q + \frac{\partial q}{\partial t} = 0, \tag{3.5}$$

$$a\frac{\partial q}{\partial y} - \beta\frac{\partial}{\partial y}(y\ p) + y\frac{\partial p}{\partial x} + \frac{\partial p}{\partial t} = 0. \tag{3.6}$$

Two differentiations of Eq. (3.5) with respect to y and the use of Eq. (3.6) yield

$$\begin{aligned} \frac{\partial}{\partial y}\Bigg[& \frac{\partial^2 p}{\partial t^2} + 2y\frac{\partial^2 p}{\partial x\partial t} + y^2\frac{\partial^2 p}{\partial x^2} + (\beta^2y^2-a^2)\frac{\partial^2 p}{\partial y^2} \\ & -2\beta y\frac{\partial^2 p}{\partial t\partial y} - 2\beta y^2\frac{\partial^2 p}{\partial y\partial x} + (2\lambda-3\beta)\frac{\partial p}{\partial t} \\ & +2(\lambda-2\beta)y\frac{\partial p}{\partial x} \\ & -2\beta(\lambda-2\beta)y\frac{\partial p}{\partial y} - 2\beta(\lambda-\beta)p\Bigg] \\ & +y\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial x\partial t} = 0. \end{aligned} \tag{3.7}$$

Equation (3.7) is a third-order partial differential equation with three independent variables that we will later reduce to an ordinary differential equation of second order. Before we note that in the Gaussian white-noise limit

$$a\to\infty, \qquad \lambda\to\infty, \qquad D\equiv\frac{a^2}{\lambda}<\infty,$$

Eq. (3.7) reduces to the well-known Kramers equation

$$\frac{\partial p}{\partial t} = -y\frac{\partial p}{\partial x} + \beta\frac{\partial}{\partial y}(yp) + \frac{1}{2}D\frac{\partial^2 p}{\partial y^2}, \tag{3.8}$$

which corresponds to the random process

$$\ddot{X} + \beta\dot{X} = \eta(t), \tag{3.9}$$

where $\eta(t)$ is Gaussian white noise. Moreover, for the undamped case $\beta \equiv 0$, Eq. (3.7) goes to the following equation:

$$\frac{\partial}{\partial y}\left[\frac{\partial^2 p}{\partial t^2} + 2y\frac{\partial^2 p}{\partial t \partial x} + y^2\frac{\partial^2 p}{\partial x^2} - a^2\frac{\partial^2 p}{\partial y^2} + 2\lambda\frac{\partial p}{\partial t} + 2\lambda y\frac{\partial p}{\partial x}\right] + \frac{\partial^2 p}{\partial t \partial x} + y\frac{\partial^2 p}{\partial x^2} = 0, \tag{3.10}$$

which agrees with our previous result [5].

In order to reduce Eq. (3.7) to a second-order ordinary differential equation we first perform the double Fourier transform:

$$\tilde{p}(\omega, \mu; t) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy e^{-i(\omega x + \mu y)} p(x, y; t)$$

and then define new independent variables τ and ξ given by

$$\tau = \beta t - \ln\left(1 - \frac{\beta\mu}{\omega}\right), \qquad \xi = \frac{\beta\mu}{\omega}. \tag{3.11}$$

The resulting equation for $\tilde{p}(\omega, \mu; t)$ is (see Appendix B for details)

$$(1-\xi)^2\frac{\partial^2\tilde{p}}{\partial\xi^2} - \left(r + \frac{1}{\xi}\right)(1-\xi)\frac{\partial\tilde{p}}{\partial\xi} + \sigma^2\xi^2\tilde{p} = 0, \tag{3.12}$$

where

$$r \equiv \frac{2\lambda}{\beta}, \qquad \sigma \equiv \frac{a\omega}{\beta^2}. \tag{3.13}$$

In Appendix B we also show that the initial conditions to be satisfied by $\tilde{p}(\omega, \xi; \tau)$ are

$$\tilde{p}(\omega, \xi = \overline{\xi}; \tau) = e^{-i\omega(x_0 + y_0\overline{\xi}/\beta)}, \tag{3.14}$$

$$\frac{\partial}{\partial\xi}\tilde{p}(\omega, \xi = \overline{\xi}; \tau) = 0, \tag{3.15}$$

where

$$\overline{\xi} = 1 - e^{-\tau}. \tag{3.16}$$

We have thus reduced the problem of finding the joint density of the process to solving a linear second-order ordinary differential equation with initial conditions. In the following sections we will obtain two special and relevant solutions to the problem.

IV. THE MARGINAL PROBABILITY DENSITY OF THE VELOCITY

The marginal probability density of the velocity is defined by

$$p(y,t) = \int_{-\infty}^{\infty} p(x, y; t) dx, \tag{4.1}$$

where $p(x, y; t)$ is the joint density of the second-order process. This marginal density is the probability density function of the first-order process

$$\dot{Y} + \beta Y = F(t). \tag{4.2}$$

In the case of first-order processes driven by dichotomous Markov noise

$$\dot{Y} = f(Y) + F(t) \tag{4.3}$$

Horsthemke and Lefever [8] obtained a rather complicated integro-differential equation for the probability density function $p(y,t)$ of the process. In the driftless case $[f(Y) \equiv 0]$ this integro-differential equation reduces to the telegrapher's equation [5,8]. They also obtained the explicit expression for the stationary probability density $p_s(y)$ of process (4.3). In the case of a linear drift this stationary distribution had been previously obtained first by Wonham and Fuller [9] and later by Pawula [10] and Klyatskin [11].

From the formalism presented in Sec. III we will now readily derive a second-order partial differential equation for $p(y,t)$. This equation, which has a structure that resembles the telegrapher's equation with variable coefficients, is equivalent to the one derived by Sancho [12] using a different approach. In [12] an exact expression for $p(y,t)$ was also obtained. Nevertheless, this expression is of such a mathematical complexity that only very few results can be extracted from it. Notwithstanding that we will obtain an exact solution for the characteristic function $\tilde{p}(\mu, t)$ [i.e., the Fourier transform of $p(y,t)$] since it contains the same information as $p(y,t)$ with the advantage of having a simpler expression that clearly shows the critical behavior of the first-order process (4.2).

A. The equation for $p(y,t)$

We start our derivation from Eqs. (3.5) and (3.6) that combined with Eq. (4.1) yield

$$a\frac{\partial p}{\partial y} - \beta\frac{\partial}{\partial y}(y\ q) + 2\lambda q + \frac{\partial q}{\partial t} = 0, \tag{4.4}$$

$$a\frac{\partial q}{\partial y} - \beta\frac{\partial}{\partial y}(y\ p) + \frac{\partial p}{\partial t} = 0, \tag{4.5}$$

where now $p = p(y,t)$ is the marginal density (4.1) and $q = q(y,t)$ is the marginal auxiliary function [cf. Eq. (3.4)]:

$$q(y,t) = \int_{-\infty}^{\infty} q(x, y; t) dx. \tag{4.6}$$

The differentiation of Eq. (4.4), the use of Eq. (4.5), and some reorganization of terms finally yield

$$\frac{\partial^2 p}{\partial t^2} + (2\lambda - \beta)\frac{\partial p}{\partial t} = \frac{\partial}{\partial y}\left[(a^2 - \beta^2 y^2)\frac{\partial p}{\partial y} + 2\beta(\lambda - \beta)yp + 2\beta y \frac{\partial p}{\partial t}\right]. \qquad (4.7)$$

This equation agrees with previous results [12].

Let us now find the initial conditions attached to Eq. (4.7). The first initial condition is obviously given by

$$p(y,0) = \delta(y - y_0). \qquad (4.8)$$

On the other hand, we know that [cf. Eqs. (3.3), (3.4), and (4.6)]

$$q(y,0) = 0.$$

The substitution of these two conditions into Eq. (4.5) immediately leads to the second initial condition

$$\left.\frac{\partial p(y,t)}{\partial t}\right|_{t=0} = \beta y_0 \delta'(y - y_0), \qquad (4.9)$$

where the prime denotes the derivative with respect to the argument.

Note that for the undamped case ($\beta \equiv 0$) Eq. (4.7) reduces to the telegrapher's equation

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda\frac{\partial p}{\partial t} = a^2 \frac{\partial^2 p}{\partial y^2}. \qquad (4.10)$$

We also observe that in the Gaussian white-noise limit Eq. (4.7) reduces to the Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = \beta\frac{\partial}{\partial y}(y\, p) + \frac{1}{2}D\frac{\partial^2 p}{\partial y^2}. \qquad (4.11)$$

B. Stationary distribution and asymptotic behavior

As we have mentioned in Sec. I the process (4.1) has a stationary distribution, $p_s(y)$, when $t \to \infty$. In order to obtain the equation satisfied by $p_s(y)$ we write Eq. (4.7) in the form

$$\frac{\partial^2}{\partial y^2}\left[(a^2 - \beta^2 y^2)p\right] + 2\lambda\beta\frac{\partial}{\partial y}(y\, p) = \frac{\partial^2 p}{\partial t^2} + (2\lambda - 3\beta)\frac{\partial p}{\partial t} - 2\beta y \frac{\partial^2 p}{\partial t \partial y}. \qquad (4.12)$$

In the stationary regime $p(y,t)$ is independent of t and Eq. (4.12) reduces to a second-order ordinary differential equation for $p_s(y)$:

$$\frac{d^2}{dy^2}\left[(a^2 - \beta^2 y^2)p_s(y)\right] + 2\lambda\beta\frac{d}{dy}\left[y\, p_s(y)\right] = 0. \qquad (4.13)$$

This equation can be integrated once with the result

$$\frac{d}{dy}\left[(a^2 - \beta^2 y^2)p_s(y)\right] + 2\lambda\beta y p_s(y) = C, \qquad (4.14)$$

where C is a constant of integration. If we assume that $C \equiv 0$ then the normalized solution of (4.14) is

$$p_s(y) = \begin{cases} \dfrac{\beta}{a}\dfrac{(1 - \beta^2 y^2/a^2)^{-1+\lambda/\beta}}{2^{-1+2\lambda/\beta}B(\lambda/\beta, \lambda/\beta)} & \text{if } -a/\beta < y < a/\beta \\ 0 & \text{otherwise}, \end{cases} \qquad (4.15)$$

where $B(z,z)$ is the beta function. Equation (4.15) agrees with previous results [9,11,10,12].

In Fig. 1 we show the different behavior of $p_s(y)$ in the cases (i) $\lambda > \beta$ and (ii) $\lambda < \beta$. In case (i) the stationary density (4.15) vanishes at the fixed points $y = \pm a/\beta$ and attains its maximum value at $y = 0$. Nevertheless, in case (ii) $p_s(y)$ diverges at $y = \pm a/\beta$ and the probability of finding the process in a neighborhood of the fixed points is very large while it is precisely at $y = 0$ where now $p_s(y)$ attains its minimum value. To get a deeper insight into this kind of "phase-transition" behavior we will study the "approach to the equilibrium," in other words, the asymptotic behavior of $p(y,t)$ as time increases.

In order to obtain the asymptotic behavior of $p(y,t)$ we follow a reasoning based on the Tauberian theorems [13]. We first Laplace transform Eq. (4.7) to get

$$\frac{\partial^2}{\partial y^2}\left[(a^2 - \beta^2 y^2)\hat{p}\right] + 2\beta(\lambda + s)\frac{\partial}{\partial y}(y\, \hat{p}) = s(s + 2\lambda - \beta)\hat{p} - (s + 2\lambda - \beta)\delta(y - y_0) + \beta y_0 \delta'(y - y_0), \qquad (4.16)$$

where $\hat{p} = \hat{p}(y,s)$ is the Laplace transform of $p(y,t)$. The asymptotic (in time) behavior of $p(y,t)$ may then be obtained by passing to the limit $s \to 0$ in Eq. (4.16). The small s approximation is found in this way to be

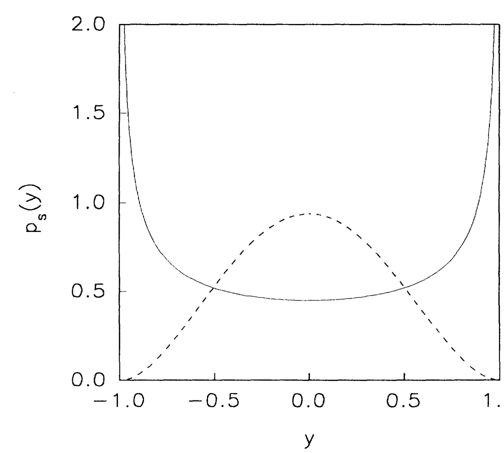

FIG. 1. The stationary probability density $p_s(y)$ of the velocity. Parameter values: $a = 1$, $\beta = 1$, $\lambda = 0.5$ (solid line), and $\lambda = 3$ (dashed line).

$$\frac{\partial^2}{\partial y^2}\left[(a^2-\beta^2y^2)\hat{p}\right]+2\beta\lambda\frac{\partial}{\partial y}(y\,\hat{p})$$
$$=s(2\lambda-\beta)\hat{p}-(2\lambda-\beta)\delta(y-y_0)$$
$$+\beta y_0\delta'(y-y_0). \qquad (4.17)$$

The Laplace inversion of Eq. (4.17) reads

$$\frac{\partial^2}{\partial y^2}\left[(a^2-\beta^2y^2)p\right]+2\beta\lambda\frac{\partial}{\partial y}(y\,p)=(2\lambda-\beta)\frac{\partial p}{\partial t}. \quad (4.18)$$

In writing this equation we have neglected the term $\beta y_0\delta'(y-y_0)\delta(t)$ because it does not contribute to the asymptotic behavior.

We observe that when $2\lambda>\beta$ Eq. (4.18) is a diffusion equation with a state-dependent diffusion coefficient given by

$$D(y)=\frac{a^2-\beta^2y^2}{\lambda-\beta/2}.$$

Recalling that $1/2\lambda=\tau_c$ is the correlation time of the dichotomous noise and that $1/\beta=\tau_r$ is the relaxation time of the process we see that when $\tau_r>\tau_c$ the system approaches the equilibrium through a diffusion process. Nevertheless, when $\tau_r<\tau_c$ Eq. (4.18) is not a diffusion equation. In this case the system goes to equilibrium following a way which does not resemble at all that of diffusion processes. This apparently surprising behavior has a rather simple physical explanation, because when $\beta>2\lambda$ the probability $e^{-\lambda t}$, that the noise $F(t)$ has not changed its value at time t, is larger than the quantity $e^{-\beta t}$ which quickly becomes very small. Therefore the random process $Y(t)$ relaxes towards one of the fixed points $\pm a/\beta$ [cf. Eq. (2.3)],

$$Y(t)\sim\pm\frac{a}{\beta},$$

in such a fast way that the noise has no time to switch its value. As a result, we have a probability close to 1 of finding the system in the neighborhood of $\pm a/\beta$. This kind of behavior is totally oposed to that of the diffusion processes.

We finally mention that the cases $\lambda=\beta$ and $\beta<\lambda<2\beta$ are singular since for the former the system has a uniform stationary distribution [cf. Eq. (4.15)]

$$p_s(y)=\begin{cases}\dfrac{\beta}{2a} & \text{if } -a/\beta<y<a/\beta\\ 0 & \text{otherwise,}\end{cases} \qquad (4.19)$$

while for the latter the stationary distribution has infinite derivatives at the fixed points (Fig. 2).

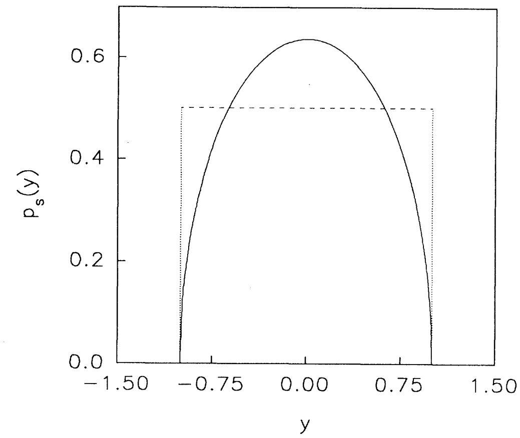

FIG. 2. Same as in Fig. 1 with $\lambda=1.5$ (solid line) and $\lambda=1$ (dashed line).

C. The characteristic function

We will now find an exact expression for the characteristic function

$$\tilde{p}(\mu,t)=\int_{-\infty}^{\infty}dye^{-i\mu y}p(y,t) \qquad (4.20)$$

of the process (4.2). In Appendix C we show that in the variables

$$\tau=\beta t-\ln\left(\frac{a\mu}{\beta}\right),\qquad \xi=\frac{a\mu}{\beta}, \qquad (4.21)$$

the characteristic function $\tilde{p}(\mu,t)$ obeys the ordinary differential equation

$$\xi\frac{d^2\tilde{p}}{d\xi^2}+\frac{2\lambda}{\beta}\frac{d\tilde{p}}{d\xi}+\xi\tilde{p}=0, \qquad (4.22)$$

with the initial conditions

$$\tilde{p}(\overline{\xi},\tau)=e^{-i\beta y_0\overline{\xi}/a}, \qquad (4.23)$$

$$\frac{d}{d\xi}\tilde{p}(\xi,\tau)\bigg|_{\xi=\overline{\xi}}=0, \qquad (4.24)$$

where

$$\overline{\xi}=e^{-\tau}. \qquad (4.25)$$

If we define the function

$$\tilde{\Phi}(\xi,\tau)=\xi^{-\alpha}\tilde{p}(\xi,\tau), \qquad (4.26)$$

where

$$\alpha=\frac{1}{2}-\frac{\lambda}{\beta}, \qquad (4.27)$$

then $\tilde{\Phi}(\xi,\tau)$ obeys the following Bessel equation:

$$\xi^2\frac{d^2\tilde{\Phi}}{d\xi^2}+\xi\frac{d\tilde{\Phi}}{d\xi}+\left(\xi^2-\alpha^2\right)\tilde{\Phi}=0, \qquad (4.28)$$

whose general solution is

$$\tilde{\Phi}(\xi,\tau)=A(\tau)J_\nu(\xi)+B(\tau)Y_\nu(\xi), \qquad (4.29)$$

where J_ν and Y_ν are Bessel functions and

$$\nu = \begin{cases} \alpha & \text{if } 2\lambda < \beta \\ -\alpha & \text{if } 2\lambda > \beta. \end{cases} \tag{4.30}$$

The constants of integration A and B are easily determined from the initial conditions. In terms of the original variables the final solution reads

$$\tilde{p}(\mu,t) = \frac{\pi}{2}\left(\frac{a\mu}{\beta}\right) e^{-(2\lambda+\beta)t/2}\left[-Y_{\frac{\lambda}{\beta}+\frac{1}{2}}\left(\frac{a\mu}{\beta}e^{-\beta t}\right) J_{\frac{\lambda}{\beta}-\frac{1}{2}}\left(\frac{a\mu}{\beta}\right) + J_{\frac{\lambda}{\beta}+\frac{1}{2}}\left(\frac{a\mu}{\beta}e^{-\beta t}\right) Y_{\frac{\lambda}{\beta}-\frac{1}{2}}\left(\frac{a\mu}{\beta}\right)\right] e^{-i\mu y_0 e^{-\beta t}}, \tag{4.31}$$

if $2\lambda > \beta$, and

$$\tilde{p}(\mu,t) = \frac{\pi}{2}\left(\frac{a\mu}{\beta}\right) e^{-(2\lambda+\beta)t/2}\left[Y_{-\frac{\lambda}{\beta}-\frac{1}{2}}\left(\frac{a\mu}{\beta}e^{-\beta t}\right) J_{-\frac{\lambda}{\beta}+\frac{1}{2}}\left(\frac{a\mu}{\beta}\right) - J_{-\frac{\lambda}{\beta}-\frac{1}{2}}\left(\frac{a\mu}{\beta}e^{-\beta t}\right) Y_{-\frac{\lambda}{\beta}+\frac{1}{2}}\left(\frac{a\mu}{\beta}\right)\right] e^{-i\mu y_0 e^{-\beta t}}, \tag{4.32}$$

if $2\lambda < \beta$.

Equations (4.31) and (4.32) furnish the complete solution to the problem and show the different behavior of the system depending on the values of the correlation time $2\lambda^{-1}$ and the relaxation time β^{-1}. Let us now find the asymptotic (in time) behavior of the characteristic function. We observe that when t is large the quantity

$$z = \left|\frac{a\mu}{\beta}\right| e^{-\beta t}$$

is small. Then, as $z \to 0$, we have the approximations [14]

$$J_\nu(z) \sim \frac{1}{\Gamma(1+\nu)}\left(\frac{z}{2}\right)^\nu$$

and

$$Y_\nu(z) \sim \begin{cases} \dfrac{-\Gamma(\nu)}{\pi}\left(\dfrac{z}{2}\right)^{-\nu} & \text{if } \nu > 0 \\ \dfrac{\cot \nu\pi}{\Gamma(1+\nu)}\left(\dfrac{z}{2}\right)^{\nu} & \text{if } \nu < 0 \end{cases}$$

provided that ν is not an integer. Introducing these approximations into Eqs. (4.31) and (4.32) we get for the stationary characteristic function, defined by

$$\tilde{p}_s(\mu) = \lim_{t\to\infty} \tilde{p}(\mu,t),$$

the following expressions:

$$\tilde{p}_s(\mu) = \Gamma\left(\frac{1}{2}+\frac{\lambda}{\beta}\right)\left(\frac{a\mu}{2\beta}\right)^{\frac{1}{2}-\frac{\lambda}{\beta}} J_{-\frac{1}{2}+\frac{\lambda}{\beta}}\left(\frac{a\mu}{\beta}\right) \tag{4.33}$$

if $2\lambda > \beta$, and

$$\tilde{p}_s(\mu) = \Gamma\left(\frac{1}{2}+\frac{\lambda}{\beta}\right)\left(\frac{a\mu}{2\beta}\right)^{\frac{1}{2}-\frac{\lambda}{\beta}} \times\left[\cos\left(\frac{1}{2}-\frac{\lambda}{\beta}\right)\pi\ J_{\frac{1}{2}-\frac{\lambda}{\beta}}\left(\frac{a\mu}{\beta}\right) - \sin\left(\frac{1}{2}-\frac{\lambda}{\beta}\right)\pi\ Y_{\frac{1}{2}-\frac{\lambda}{\beta}}\left(\frac{a\mu}{\beta}\right)\right] \tag{4.34}$$

if $2\lambda < \beta$.

By means of the confluent hypergeometric function $F(\alpha,\gamma;z)$ [14] these two expressions can be written in the following single expression valid for all values of λ and β:

$$\tilde{p}_s(\mu) = e^{-ia\mu/\beta} F\left(\frac{\lambda}{\beta},\frac{2\lambda}{\beta};2i\frac{a\mu}{\beta}\right). \tag{4.35}$$

The Fourier inversion of Eq. (4.35) reads [15]

$$p_s(y) = \begin{cases} \dfrac{\beta}{a}\,\dfrac{(1-\beta^2y^2/a^2)^{-1+\lambda/\beta}}{2^{-1+2\lambda/\beta}B(\lambda/\beta,\lambda/\beta)} & \text{if } -a/\beta < y < a/\beta \\ 0 & \text{otherwise}, \end{cases} \tag{4.36}$$

which is precisely Eq. (4.15).

V. THE MARGINAL PROBABILITY DENSITY OF THE POSITION

In terms of the joint density $p(x,y;t)$ the marginal density of $X(t)$ is given by

$$p(x,t) = \int_{-\infty}^{\infty} p(x,y;t)dy \tag{5.1}$$

and its Fourier transform is

$$\tilde{p}(\omega,t) = \tilde{p}(\omega,\rho=0;t), \tag{5.2}$$

where $\tilde{p}(\omega,\rho;t)$ is the joint Fourier transform of $p(x,y;t)$.

The evaluation of $p(x,t)$ for inertial processes is usually very involved [2,5,18] and there are few cases where exact equations can be obtained. The reason for this difficulty lies mainly in the fact that the position at a given time is strongly dependent on the velocity and to get rid of this dependence becomes almost impossible in many cases. In spite of these technical difficulties in [5] we have obtained, for the undamped case, an exact equation for $p(x,t)$. In this section we will use an analogous technique to obtain this marginal density for our damped process.

A. Equation for $p(x,t)$

Let us now derive an exact equation for the marginal density $p(x,t)$. The starting point of our derivation is Eq. (3.12) that we write in the form

$$(1-\xi)^2\tilde{p}'' - \left(r+\frac{1}{\xi}\right)(1-\xi)\tilde{p}' + \sigma^2\xi^2\tilde{p} = 0, \quad (5.3)$$

where $\tilde{p} = \tilde{p}(\omega,\mu,\tau)$ is the joint characteristic function, ξ and τ are defined by Eq. (3.11), r and σ by Eq. (3.13), and the primes denote derivatives with respect to ξ. The initial conditions for Eq. (5.3) are [cf. Eqs. (3.14) and (3.15)]

$$\tilde{p}(\omega,\xi=\overline{\xi};\tau) = e^{-i\omega(x_0+y_0\overline{\xi}/\beta)}, \quad (5.4)$$

$$\tilde{p}'(\omega,\xi=\overline{\xi};\tau) = 0, \quad (5.5)$$

where $\overline{\xi} = 1-e^{-\tau}$.

It is shown in Appendix D that the solution to the problem (5.3)–(5.5), in the original variables μ and t, is

$$\dot{p}(\omega,\mu,t) = \frac{1}{2}\frac{\left(1-\frac{\beta\mu}{\omega}\right)^{1+r} e^{-(1+r)\beta t}}{1-\left(1-\frac{\beta\mu}{\omega}\right)e^{-\beta t}}\left\{u\left(\frac{\beta\mu}{\omega}\right)v'\left[1-\left(1-\frac{\beta\mu}{\omega}\right)e^{-\beta t}\right] - v\left(\frac{\beta\mu}{\omega}\right)u'\left[1-\left(1-\frac{\beta\mu}{\omega}\right)e^{-\beta t}\right]\right\}$$
$$\times\exp\left(-i\omega\left\{x_0+\frac{y_0}{\beta}\left[1-\left(1-\frac{\beta\mu}{\omega}\right)e^{-\beta t}\right]\right\}\right), \quad (5.6)$$

where $u(\xi)$ and $v(\xi)$ are two independent solutions of Eq. (5.3) such that

$$u(\xi) = 1+O(\xi^4), \qquad v(\xi) = \xi^2 + \tfrac{2}{3}(1+r)\xi^3 + O(\xi^4). \quad (5.7)$$

Now setting $\mu = 0$ and taking into account Eq. (5.7) we obtain the marginal characteristic function of $X(t)$

$$\tilde{p}(\omega;t) = \frac{1}{2}\frac{e^{-(1+r)\beta t}}{1-e^{-\beta t}}\,v'\left(1-e^{-\beta t}\right)$$
$$\times\exp\left\{-i\omega\left[x_0+\frac{y_0}{\beta}(1-e^{-\beta t})\right]\right\}. \quad (5.8)$$

Let us now find a closed equation for $p(x,t)$. We first note that the function $v(\zeta)$, where

$$\zeta = \zeta(t) \equiv 1-e^{-\beta t}, \quad (5.9)$$

is a solution to Eq. (5.3), that is,

$$(1-\zeta)^2 v'' - \left(r+\frac{1}{\zeta}\right)(1-\zeta)v' + \sigma^2\zeta^2 v = 0. \quad (5.10)$$

If we differentiate Eq. (5.10) with respect to ζ and use again Eq. (5.10) to write v in terms of v' and v'' we get the following equation for v':

$$(1-\zeta)^2\frac{d^2v'}{d\zeta^2} - (1-\zeta)\left(r+\frac{3}{\zeta}\right)\frac{dv'}{d\zeta}$$
$$+\left[\sigma^2\zeta^2 - r + \frac{2(r-1)}{\zeta} + \frac{3}{\zeta^2}\right]v' = 0. \quad (5.11)$$

We define the auxiliary function

$$\tilde{\Phi}(\omega,t) \equiv e^{i\omega y_0\zeta(t)/\beta}\tilde{p}(\omega,t), \quad (5.12)$$

or in real space

$$\Phi(x,t) = p\left(x+\frac{y_0}{\beta}\zeta(t),t\right). \quad (5.13)$$

The substitution of Eq. (5.8) along with Eq. (5.12) into Eq. (5.11) results in the following equation for $\tilde{\Phi}(\omega,\zeta)$:

$$(1-\zeta)^2\frac{\partial^2\tilde{\Phi}}{\partial\zeta^2} + (1-\zeta)\left(r-\frac{1}{\zeta}\right)\frac{\partial\tilde{\Phi}}{\partial\zeta} + \sigma^2\zeta^2\tilde{\Phi} = 0,$$

or in the original time scale:

$$\frac{\partial^2\tilde{\Phi}}{\partial t^2} + \left(2\lambda - \beta\,\frac{e^{-\beta t}}{1-e^{-\beta t}}\right)\frac{\partial\tilde{\Phi}}{\partial t}$$
$$+\frac{a^2\omega^2}{\beta^2}\left(1-e^{-\beta t}\right)^2\tilde{\Phi} = 0. \quad (5.14)$$

Finally the Fourier inversion of Eq. (5.14) yields [cf. Eqs. (5.9) and (5.13)]

$$\left[\frac{\partial^2}{\partial t^2} + \left(2\lambda - \beta\,\frac{e^{-\beta t}}{1-e^{-\beta t}}\right)\frac{\partial}{\partial t}\right]p\left(x+\frac{y_0}{\beta}(1-e^{-\beta t}),\ t\right)$$
$$= \frac{a^2}{\beta^2}(1-e^{-\beta t})^2\frac{\partial^2}{\partial x^2}p\left(x+\frac{y_0}{\beta}(1-e^{-\beta t}),\ t\right). \quad (5.15)$$

This is an exact equation for the marginal density $p(x,t)$ of $X(t)$ with the position, x, shifted to $x+(y_0/\beta)(1-e^{-\beta t})$. It is a relatively simple equation having the structure of the telegrapher's equation with time varying coefficients. Nevertheless, the equation for $p(x,t)$ is more complicated and we will not write it.

We can easily see that in the undamped case, $\beta \equiv 0$, Eq. (5.15) reduces to the equation

$$\left[\frac{\partial^2}{\partial t^2} + \left(2\lambda - \frac{1}{t}\right)\frac{\partial}{\partial t}\right]p(x+y_0t,t)$$
$$= a^2t^2\frac{\partial^2}{\partial x^2}p(x+y_0t,t), \quad (5.16)$$

which, when $y_0 = 0$, agrees with our previous result [5].

We now give the initial conditions that accompany Eq. (5.15). The first one is obviously given by

$$p\left(x + \frac{y_0}{\beta}(1 - e^{-\beta t}),\ t\right)\Big|_{t=0} = p(x,0) = \delta(x - x_0). \tag{5.17}$$

In order to find the second initial condition we write [cf. Eqs. (5.8), (5.9), and (5.12)]

$$\tilde{\Phi}(\omega, \zeta(t)) = \frac{[1 - \zeta(t)]^{1+r}}{2\,\zeta(t)}\, v'(\zeta(t)),$$

thus

$$\frac{\partial \tilde{\Phi}}{\partial t} = \beta\, \frac{(1-\zeta)^{1+r}}{2\,\zeta}\left[(1-\zeta)\left(v''(\zeta) - \frac{1}{\zeta}v'(\zeta)\right) - (1+r)v'(\zeta)\right]. \tag{5.18}$$

Taking into account that [cf. Eq. (5.7)]

$$v'(\zeta) = 2\zeta + 2(1+r)\zeta^2 + O(\zeta^3)$$

and noting that $t = 0$ is equivalent to setting $\zeta = 0$ we see from Eq. (5.18) that

$$\frac{\partial}{\partial t}\tilde{\Phi}(\omega, t)\Big|_{t=0} = 0.$$

Therefore the second initial condition for Eq. (5.15) reads

$$\frac{\partial}{\partial t}\, p\left(x + \frac{y_0}{\beta}(1 - e^{-\beta t}),\ t\right)\Big|_{t=0} = 0. \tag{5.19}$$

B. The asymptotic behavior

We will now perform the asymptotic analysis of Eq. (5.15) in order to get a greater insight into how $p(x,t)$ evolves in time.

As in the case of the marginal distribution of the velocity, there are two time scales involved in the process, namely, the time scale given by β^{-1} that we call "relaxation time" because it represents the relaxation time of the velocity $\dot{X}(t)$ and the correlation time of the noise $(2\lambda)^{-1}$. We will now show that $p(x,t)$ has a different asymptotic behavior depending on whether the observation time t is greater than one (or both) of these time scales.

We first assume that t is much greater than the relaxation time and that the correlation time is arbitrary. Thus $\beta t \gg 1$ and Eq. (5.15) is approximated by

$$\left[\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t}\right] p\left(x + \frac{y_0}{\beta}, t\right) = \frac{a^2}{\beta^2}\frac{\partial^2}{\partial x^2} p\left(x + \frac{y_0}{\beta}, t\right). \tag{5.20}$$

We observe that Eq. (5.20) is an ordinary telegrapher's equation and corresponds to a free first-order dichotomous process. Indeed, if $\beta t \gg 1$ then Eqs. (2.3) and (2.1) read

$$y^{\pm}(t; y_0) \simeq \pm\frac{a}{\beta},$$

$$x^{\pm}(t; x_0, y_0) \simeq x_0 + \frac{y_0}{\beta} \pm \frac{a}{\beta}t,$$

but this corresponds to the first-order dichotomous process:

$$\dot{X}(t) = \beta^{-1}F(t),$$

with $X(0) = x_0 + y_0/\beta$ and the probability density function of $X(t)$ obeys the telegrapher's equation (5.20). Moreover, if the observation time t is also much greater than the correlation time $(2\lambda)^{-1}$ then the dominant balance technique implies [23,5]

$$\left|\frac{\partial^2 p}{\partial t^2}\right| \ll 2\lambda \left|\frac{\partial p}{\partial t}\right| \tag{5.21}$$

and Eq. (5.20) reduces to the diffusion equation

$$\frac{\partial p}{\partial t} = \frac{a^2}{2\lambda\beta^2}\,\frac{\partial^2 p}{\partial x^2}, \tag{5.22}$$

which is simply the Smoluchowski equation for the free Brownian motion (see next section).

We now assume that the observation time is much greater than the correlation time and that the relaxation time is arbitrary. Let us now write Eq. (5.14) in the form

$$\left[(1 - e^{-\beta t})\left(\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t}\right) - \beta e^{-\beta t}\frac{\partial}{\partial t} + \frac{a^2\omega^2}{\beta^2}(1 - e^{-\beta t})^3\right]\tilde{\Phi}(\omega, t) = 0$$

but if $t \gg (2\lambda)^{-1}$ then the dominant balance technique [cf. Eq. (5.21)] applied to this equation yields

$$\left[\left(2\lambda(1 - e^{-\beta t}) - \beta e^{-\beta t}\frac{\partial}{\partial t}\right) + \frac{a^2\omega^2}{\beta^2}(1 - e^{-\beta t})^3\right]\tilde{\Phi}(\omega, t) = 0$$

and, after inverting the Fourier transform, we obtain the diffusion equation

$$\left[\frac{\partial}{\partial t} - D(t)\frac{\partial^2}{\partial x^2}\right] p\left(x + \frac{y_0}{\beta}(1 - e^{-\beta t}),\ t\right) = 0, \tag{5.23}$$

with a variable diffusion coefficient given by

$$D(t) = \frac{a^2}{\beta^2} \frac{(1-e^{-\beta t})^3}{2\lambda(1-e^{-\beta t}) - \beta e^{-\beta t}}, \qquad t \gg (2\lambda)^{-1}. \tag{5.24}$$

In this case we have only assumed that the observation time t is much greater than the correlation time and that the relaxation time is arbitrary. Depending on how the relaxation time is we have two different situations. Thus if $t \gg \beta^{-1}$ we get from Eq. (5.24)

$$D(t) \sim \frac{a^2}{2\lambda\beta^2}$$

and Eq. (5.23) reduces to the Smoluchowski equation (5.22). On the other hand, if $t \ll \beta^{-1}$ we have from Eq. (5.24)

$$D(t) \sim \frac{a^2t^2}{2\lambda}$$

and Eq. (5.23) agrees with our previous result on the undamped case.

We now summarize the asymptotic results. It follows from the preceding analysis that the orders of magnitude of the correlation time $(2\lambda)^{-1}$ and the relaxation time β^{-1} are crucial for the time evolution of $p(x,t)$. When both orders of magnitude are similar we see from above that the asymptotic evolution of $p(x,t)$ is given by the Smoluchowski equation (5.22). Nevertheless, the evolution can be very different when both orders are not comparable. In this situation two cases are involved.

(1) The correlation time is much greater than the relaxation time,

$$(2\lambda)^{-1} \gg \beta^{-1}.$$

In this case we distinguish two different time regimes. (a) The observation time is much greater than the relaxation time but still smaller than the correlation time:

$$\beta^{-1} \ll t < (2\lambda)^{-1}.$$

In this regime $p(x,t)$ obeys the ordinary telegrapher's equation (5.20). The solution to this equation can be found in [5]. Therefore the time evolution of $p(x,t)$ is approximately given by

$$\begin{aligned} p(x,t/x_0,y_0) \sim \frac{1}{2}e^{-\lambda t}\Bigg\{ & \delta(at - |x-x_0-y_0/\beta|) \\ & + \lambda\left[(1/a)I_0\left(\frac{\lambda}{a}\sqrt{a^2t^2-(x-x_0-y_0/\beta)^2}\right)\right. \\ & \left. + \frac{t}{\sqrt{a^2t^2-(x-x_0-y_0/\beta)^2}} I_1\left(\frac{\lambda}{a}\sqrt{at^2-(x-x_0-y_0/\beta)^2}\right)\right] \\ & \times\Theta\left(at-|x-x_0-y_0/\beta|\right)\Bigg\}. \end{aligned} \tag{5.25}$$

(b) The observation time is much greater than the correlation time:

$$t \gg (2\lambda)^{-1} \gg \beta^{-1}.$$

In this regime $p(x,t)$ obeys the Smoluchowski equation (5.22) and the time evolution of $p(x,t)$ is approximately given by the Gaussian density

$$p(x,t) \sim \frac{1}{\sqrt{2\pi\rho(t)}} \exp\left\{-\frac{(x-x_0-y_0/\beta)^2}{2\rho(t)}\right\}, \tag{5.26}$$

where

$$\rho(t) = \frac{a^2t}{\lambda\beta^2}. \tag{5.27}$$

(2) The correlation time is much smaller than the relaxation time,

$$(2\lambda)^{-1} \ll \beta^{-1}.$$

Again, we distinguish two time regimes. (a) The observation time is much greater than the correlation time but still smaller than the relaxation time:

$$(2\lambda)^{-1} \ll t < \beta^{-1}.$$

In this regime $p(x,t)$ obeys the diffusion equation (5.23). Note that now $2\lambda \gg \beta$ hence the variable diffusion coefficient (5.24) can be written in the form

$$D(t) \simeq \frac{a^2}{2\lambda\beta^2}\left(1-e^{-\beta t}\right)^2.$$

Therefore the time evolution is given by the Gaussian density (5.26) where $\rho(t)$ is now given by

$$\rho(t) = \frac{a^2}{\lambda\beta^3}\left(\tfrac{1}{2}\beta t - \tfrac{3}{4} + e^{-\beta t} - \tfrac{1}{4}e^{-2\beta t}\right). \tag{5.28}$$

As we will see in the next section this case corresponds to that of Gaussian white driving noise. (b) The observation time is much greater than the relaxation time:

$$t \gg \beta^{-1} \gg (2\lambda)^{-1}.$$

In this regime $p(x,t)$ obeys the Smoluchowski equation

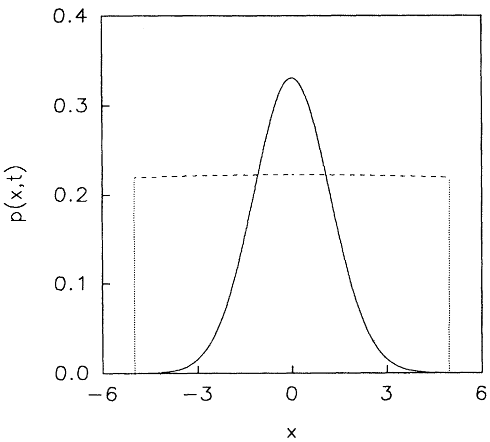

FIG. 3. Probability density of the position $p(x,t)$ at $t=5$. Parameter values: $x_0 = y_0 = 0$, $a = 1$. Solid line: $\lambda = 1$ and $\beta = 0.1$ [Gaussian density (5.26) with (5.28)]. Dashed line: $\lambda = 0.05$ and $\beta = 1$ [telegraphic process (5.25)].

(5.22) and the time evolution of $p(x,t)$ is approximately given by (5.26) and (5.27).

Figure 3 shows the different behavior of $p(x,t)$ according to the order of magnitude of the correlation time versus the relaxation time. Thus when $(2\lambda)^{-1}$ is of the same order of magnitude or smaller than β^{-1} the density $p(x,t)$ directly evolves to a Gaussian density. However, when $(2\lambda)^{-1}$ is (much) greater than β^{-1} this evolution is also towards a Gaussian density *but* through a telegraphic process.

VI. FREE BROWNIAN MOTION AND ANOMALOUS DIFFUSION

In the Gaussian white-noise limit

$$a \to \infty, \qquad \lambda \to \infty, \qquad D = \frac{a^2}{\lambda} < \infty, \tag{6.1}$$

our dichotomous Markov noise $F(t)$ tends (in probability) to Gaussian white noise $\eta(t)$ and the second-order process

$$\ddot{X} + \beta \dot{X} = \eta(t) \tag{6.2}$$

is precisely the free Brownian motion. From the formalism described above we can easily obtain the exact equation for the probability density of the displacement of a free Brownian particle. Indeed, it is straightforward to find that the Gaussian white-noise limit (6.1) turns Eq. (5.15) into the following Fokker-Planck equation:

$$\left[\frac{\partial}{\partial t} - \frac{D}{2\beta^2}(1-e^{-\beta t})^2 \frac{\partial^2}{\partial x^2}\right] p\left(x + \frac{y_0}{\beta}(1-e^{-\beta t}),\ t\right) = 0. \tag{6.3}$$

This equation has to be solved under the initial condition [cf. Eq. (5.17)]

$$p\left(x + \frac{y_0}{\beta}(1-e^{-\beta t}), t\right)\bigg|_{t=0} = p(x,0) = \delta(x - x_0). \tag{6.4}$$

Equation (6.3) is a diffusion equation with a time varying diffusion coefficient given by

$$D(t) = \frac{D}{2\beta^2}(1 - e^{-\beta t})^2. \tag{6.5}$$

We can also write Eq. (6.3) in the form

$$\left[\frac{\partial}{\partial t} + y_0 e^{-\beta t}\frac{\partial}{\partial x} - \frac{D}{2\beta^2}(1-e^{-\beta t})^2 \frac{\partial^2}{\partial x^2}\right] p(x,t) = 0. \tag{6.6}$$

It is in this form that the equation for the density of the displacement of a Brownian particle was first obtained by Mazo [16] and later by other authors using different methods [17,18].

We easily get from Eq. (6.3) two relevant asymptotic behaviors. The first one is the classic result of Smoluchowski. Thus, when the damping β is large, which is equivalent to assuming that $t \gg \beta^{-1}$, we may write

$$x + \frac{y_0}{\beta}(1 - e^{-\beta t}) \sim x + \frac{y_0}{\beta} \sim x,$$

where we also assume that $|y_0|/\beta \ll x$. In this case Eq. (6.3) reduces to the Smoluchowski equation [19–21]

$$\left[\frac{\partial}{\partial t} - \frac{D}{2\beta^2}\frac{\partial^2}{\partial x^2}\right] p(x,t) = 0. \tag{6.7}$$

The other asymptotic result refers to the low damping regime and this is equivalent to assuming that $t \ll \beta^{-1}$. In this case Eq. (6.3) reads

$$\left[\frac{\partial}{\partial t} - \frac{1}{2}Dt^2(1-\beta t)\frac{\partial^2}{\partial x^2}\right] p(x + y_0 t(1 - \tfrac{1}{2}\beta t),\ t) = 0. \tag{6.8}$$

For the undamped case $\beta \equiv 0$, we have

$$\left[\frac{\partial}{\partial t} - \frac{1}{2}Dt^2\frac{\partial^2}{\partial x^2}\right] p(x + y_0 t, t) = 0. \tag{6.9}$$

When $y_0 = 0$ this equation agrees with our previous result [5].

Let us now exactly solve Eq. (6.3). We proceed as follows. We Fourier transform Eq. (6.3) with the result

$$\left[\frac{\partial}{\partial t} + \frac{D}{2\beta^2}(1-e^{-\beta t})^2\omega^2\right] \tilde{p}(\omega,t) = 0.$$

It is straightforward to find the solution to this ordinary differential equation with the initial condition [cf. Eq. (5.17)]

$$\tilde{p}(\omega,0) = \exp\left\{-i\omega\left[x_0 + \frac{y_0}{\beta}(1-e^{-\beta t})\right]\right\}.$$

This solution is

$$\tilde{p}(\omega,t)=\exp\left\{-i\omega\left[x_0+\frac{y_0}{\beta}(1-e^{-\beta t})\right]-\frac{D\omega^2}{\beta^3}\gamma(t)\right\}, \tag{6.10}$$

where

$$\gamma(t)=\tfrac{1}{2}\beta t-\tfrac{3}{4}+e^{-\beta t}-\tfrac{1}{4}e^{-2\beta t}. \tag{6.11}$$

The Fourier inversion of (6.10) leads to the following expression that was first given by Uhlenbeck and Ornstein back in 1930 [22]:

$$p(x,t)=\left(\frac{\beta^3}{2\pi D\gamma(t)}\right)^{1/2}\times\exp\left[-\frac{\beta^3}{2D}\frac{\left[x-x_0-y_0(1-e^{-\beta t})/\beta\right]^2}{\gamma(t)}\right]. \tag{6.12}$$

A remarkable feature of both Eqs. (6.8) and (6.9) is the following anomalous superdiffusive behavior:

$$\langle X^2(t)\rangle\sim t^3,\qquad \beta t\ll 1 \tag{6.13}$$

in contrast with the ordinary diffusion behavior exhibited by the Smoluchowski equation (6.7)

$$\langle X^2(t)\rangle\sim t,\qquad \beta t\gg 1. \tag{6.14}$$

In fact, Eq. (6.13) as well as Eq. (6.14) can be obtained as limit cases of the exact variance

$$\sigma(t)=\langle X^2(t)\rangle-\langle X(t)\rangle^2$$

of the process (6.2). Indeed, from Eq. (6.12) we have

$$\sigma(t)=\frac{D}{\beta^3}(\beta t-\tfrac{3}{2}+2e^{-\beta t}-\tfrac{1}{2}e^{-2\beta t}). \tag{6.15}$$

When $\beta t\ll 1$ Eq. (6.15) goes to

$$\sigma(t)\simeq\tfrac{1}{3}Dt^3 \tag{6.16}$$

and when $\beta t\gg 1$ it goes to

$$\sigma(t)\simeq\frac{D}{\beta^2}t. \tag{6.17}$$

The transition from superdiffusion to ordinary diffusion is clearly shown by means of the "dynamical exponent," $\nu(t)$, of the process. We define this exponent in an analogous way as is done in fractal theory to define the differential fractal dimension, a quantity that characterizes the random motion of particles [24–26]:

$$\nu(t)\equiv\frac{d\ln\sigma(t)}{d\ln t}. \tag{6.18}$$

Note that in the regions where this quantity does not appreciably vary, the variance $\sigma(t)$ can be written as

$$\sigma(t)\simeq t^{\nu(t)}. \tag{6.19}$$

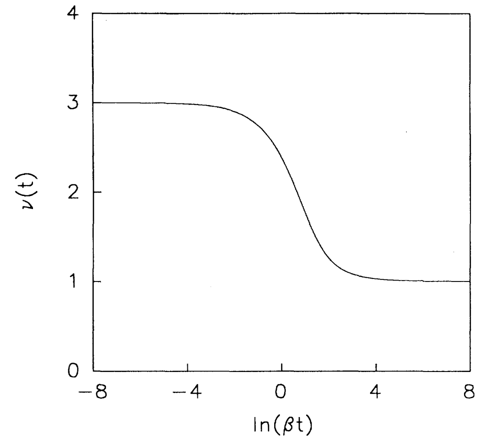

FIG. 4. Dynamical exponent of the free Brownian particle.

In our case the exponent $\nu(t)$ is given by

$$\nu(t)=\frac{\beta t(1-2e^{-\beta t}+e^{-2\beta t})}{\beta t-\frac{3}{2}+2e^{-\beta t}-\frac{1}{2}e^{-2\beta t}}. \tag{6.20}$$

In Fig. 4 we have plotted $\nu(t)$ in terms of $\ln\beta t$ and the figure clearly shows that, as time increases, there is a crossover from the dynamical exponent $\nu=3$ to $\nu=1$. We remark that the damping is responsible for this crossover since in the undamped case the dynamical exponent is always superdiffusive [5,6].

VII. CONCLUSIONS

We have studied damped inertial processes driven by dichotomous Markov noise in the absence of a potential. We have shown that the joint probability density function $p(x,y;t)$ obeys a third-order partial differential equation which, in the Fourier domain and with a convenient choice of variables, reduces to a second-order ordinary differential equation.

The marginal probability density of the velocity $p(y,t)$ obeys a second-order partial differential equation. In this case a complete solution has been provided for the characteristic function of $p(y,t)$. We have also obtained the asymptotic evolution to the stationary distribution and the stationary density itself, showing the critical behavior of the velocity depending on the values of the correlation time $\tau_c=(2\lambda)^{-1}$ and relaxation time $\tau_r=\beta^{-1}$ as can be seen from Figs. 1 and 2.

The marginal probability density of the position obeys the following telegrapher's equation with variable coefficients:

$$\left[\frac{\partial^2}{\partial t^2}+\left(2\lambda-\beta\frac{e^{-\beta t}}{1-e^{-\beta t}}\right)\frac{\partial}{\partial t}\right]p\left(x+\frac{y_0}{\beta}(1-e^{-\beta t}),t\right)=\frac{a^2}{\beta^2}(1-e^{-\beta t})^2\frac{\partial^2}{\partial x^2}p\left(x+\frac{y_0}{\beta}(1-e^{-\beta t}),t\right).$$

The asymptotic analysis of this equation also shows the totally different behavior of $p(x,t)$ depending on the values of τ_c and τ_r. Thus when $\tau_c \lesssim \tau_r$ the density directly evolves with time to a Gaussian density, while when $\tau_c \gg \tau_r$ the evolution is also towards a Gaussian density but through a telegraphic process (Fig. 3).

The Gaussian white-noise limit turns the above equation into the following exact Fokker-Planck equation for the displacement of a free Brownian particle:

$$\left[\frac{\partial}{\partial t} - \frac{D}{2\beta^2}(1-e^{-\beta t})^2 \frac{\partial^2}{\partial x^2}\right] p\left(x + \frac{y_0}{\beta}(1-e^{-\beta t}), t\right) = 0.$$

In this limit the variance $\sigma(t)$ of the process presents the superdiffusive behavior

$$\sigma(t) \sim t^3$$

if $t \ll \tau_r$, or the ordinary diffusion behavior

$$\sigma(t) \sim t$$

if $t \gg \tau_r$. This crossover effect, which is due to the damping, is clearly shown in Fig. 4.

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APPENDIX A: DERIVATION OF EQS. (3.1) AND (3.2)

With the exponential density (1.3) the time Laplace transform of the kernel $h^{\pm}(x,y;t/x_0,y_0)$ [cf. Eq. (2.7)] reads

$$\hat{h}^{\pm}(x,y;s/x_0,y_0) = \frac{\lambda}{\beta|y \mp a/\beta|}\Theta\left(\ln\left|\frac{y_0 \mp a/\beta}{y \mp a/\beta}\right|\right)\left|\frac{y \mp a/\beta}{y_0 \mp a/\beta}\right|^{(\lambda+s)/\beta} \times\delta\left(x - x_0 + \frac{1}{\beta}(y-y_0) \mp \frac{a}{\beta^2}\ln\left|\frac{y_0 \mp a/\beta}{y \mp a/\beta}\right|\right), \qquad \text{(A1)}$$

where $\Theta(z)$ is the Heaviside step function and

$$\Theta\left(\ln\left|\frac{y_0 - a/\beta}{y - a/\beta}\right|\right) = \Theta\left(y_0 - \frac{a}{\beta}\right)\Theta(y_0 - y) + \Theta\left(\frac{a}{\beta} - y_0\right)\Theta(y - y_0),$$

$$\Theta\left(\ln\left|\frac{y_0 + a/\beta}{y + a/\beta}\right|\right) = \Theta\left(y_0 + \frac{a}{\beta}\right)\Theta(y_0 - y) + \Theta\left(-\frac{a}{\beta} - y_0\right)\Theta(y - y_0).$$

We first restrict the initial velocity to be in the interval $-a/\beta < y_0 < a/\beta$ and we will later remove this restriction. In this case the final velocity lies also in this interval. Hence

$$\Theta\left(\ln\left|\frac{y_0 - a/\beta}{y - a/\beta}\right|\right) = \Theta(y - y_0)$$

and

$$\Theta\left(\ln\left|\frac{y_0 + a/\beta}{y + a/\beta}\right|\right) = \Theta(y_0 - y).$$

The x Fourier transform of $\hat{h}^{\pm}(x,y;s/x_0,y_0)$ reads

$$\hat{\hat{h}}^{\pm}(\omega,y;s/x_0,y_0) = \frac{\lambda}{\beta|y \mp a/\beta|}\Theta(\pm(y-y_0))\left|\frac{y \mp a/\beta}{y_0 \mp a/\beta}\right|^{\alpha_\pm} \exp\left\{-i\omega\left(x_0 - \frac{y-y_0}{\beta}\right)\right\}, \qquad \text{(A2)}$$

where

$$\alpha_\pm \equiv \frac{\lambda+s}{\beta} \pm i\frac{a\omega}{\beta^2}. \qquad \text{(A3)}$$

We observe that $\hat{h}^{\pm}(u,v;s/x,y) = \hat{h}^{\pm}(x-u,v;s/y)$. Therefore the joint Fourier-Laplace transform defined by

$$\hat{\hat{\Omega}}^{\pm}(\omega,y;s) = \int_0^\infty dt\, e^{-st}\int_{-\infty}^{\infty} dx\, e^{-i\omega x}\Omega^{\pm}(x,y;t)$$

turns Eqs. (2.5) and (2.6) into the following set of coupled integral equations:

$$\hat{\hat{\Omega}}^{+}(\omega,y;s)=\tfrac{1}{2}\hat{\hat{h}}^{+}(\omega,y;s)+\frac{\lambda}{\beta}\ |y-a/\beta|^{-1+\alpha_+}\int_{-a/\beta}^{y}dv\ \frac{\hat{\hat{\Omega}}^{-}(\omega,v;s)}{|v-a/\beta|^{\alpha_+}}\ e^{i\omega(y-v)/\beta}, \tag{A4}$$

$$\hat{\hat{\Omega}}^{-}(\omega,y;s)=\tfrac{1}{2}\hat{\hat{h}}^{-}(\omega,y;s)+\frac{\lambda}{\beta}\ |y+a/\beta|^{-1+\alpha_-}\int_{y}^{a/\beta}dv\ \frac{\hat{\hat{\Omega}}^{+}(\omega,v;s)}{|v+a/\beta|^{\alpha_-}}\ e^{i\omega(y-v)/\beta}. \tag{A5}$$

After differentiating these two equations with respect to y we find that the set of coupled integral equations is equivalent to the following set of first-order differential equations:

$$(a-\beta y)\frac{\partial\hat{\hat{\Omega}}^{+}}{\partial y}+(i\omega y+\lambda+s-\beta)\hat{\hat{\Omega}}^{+}-\lambda\hat{\hat{\Omega}}^{-}=\frac{\lambda}{2}\delta(y-y_0)e^{-i\omega x_0}, \tag{A6}$$

$$(a+\beta y)\frac{\partial\hat{\hat{\Omega}}^{-}}{\partial y}-(i\omega y+\lambda+s-\beta)\hat{\hat{\Omega}}^{-}+\lambda\hat{\hat{\Omega}}^{+}=-\frac{\lambda}{2}\delta(y-y_0)e^{-i\omega x_0}. \tag{A7}$$

In writing Eqs. (A6) and (A7) we have taken into account the expression of $\hat{\hat{h}}^{\pm}(\omega,y;s/x_0,y_0)$ given by Eq. (A2) and that

$$\left|y\pm\frac{a}{\beta}\right|=\frac{a}{\beta}\pm y$$

since we are assuming that $-a/\beta<y<a/\beta$. In fact, this last assumption can be removed because following the above line of reasoning it can be easily shown that Eqs. (A6) and (A7) are also valid for all values of y and y_0.

If we take into account that $\Omega^{\pm}(x,y;t)$ satisfy the initial conditions (3.3) then the Fourier-Laplace inversion of the above set of equations finally leads to the system (3.1) and (3.2).

APPENDIX B: DERIVATION OF EQ. (3.12)

The double Fourier transform of the system (3.5) and (3.6) reads

$$\left[\frac{\partial}{\partial t}-(\omega-\beta\mu)\frac{\partial}{\partial\mu}\right]\tilde{q}+2\lambda\tilde{q}+ia\mu\tilde{p}=0, \tag{B1}$$

$$\left[\frac{\partial}{\partial t}-(\omega-\beta\mu)\frac{\partial}{\partial\mu}\right]\tilde{p}+ia\mu\tilde{q}=0. \tag{B2}$$

The change of variables (3.11) turns this system into

$$(1-\xi)\frac{\partial\tilde{q}}{\partial\xi}-r\tilde{q}-i\sigma\xi\tilde{p}=0, \tag{B3}$$

$$(1-\xi)\frac{\partial\tilde{p}}{\partial\xi}-i\sigma\xi\tilde{q}=0, \tag{B4}$$

where r and σ are defined in Eq. (3.13). By combining Eqs. (B3) and (B4) we easily obtain Eq. (3.12).

Let us now obtain the initial conditions. From Eq. (3.3) we see that

$$\tilde{p}(\omega,\mu;0)=e^{-i(\omega x_0+\mu y_0)},\qquad \tilde{q}(\omega,\mu;0)=0.$$

Also from Eqs. (B2) and (3.11) we see that

$$(1-\xi)\frac{\partial}{\partial\xi}\tilde{p}\bigg|_{t=0}=0.$$

Now taking into account that $t=0$ is equivalent to setting $\xi=\overline{\xi}$, where $\overline{\xi}=1-e^{-\tau}$, we get Eqs. (3.14) and (3.15).

APPENDIX C: DERIVATION OF EQ. (4.22)

In terms of $\tilde{p}(\omega,\mu;t)$ the characteristic function $\tilde{p}(\mu;t)$ is given by

$$\tilde{p}(\mu;t)=\tilde{p}(\omega=0,\mu;t).$$

In this case the system (B1) and (B2) of Appendix B can be written as

$$\left(\frac{\partial}{\partial t}+\beta\mu\frac{\partial}{\partial\mu}\right)\tilde{q}+2\lambda\tilde{q}+ia\mu\tilde{p}=0, \tag{C1}$$

$$\left(\frac{\partial}{\partial t}+\beta\mu\frac{\partial}{\partial\mu}\right)\tilde{p}+ia\mu\tilde{q}=0. \tag{C2}$$

The change of variables (4.21) turns this system into

$$\beta\xi\frac{\partial\tilde{q}}{\partial\xi}+2\lambda\tilde{q}+i\beta\xi\tilde{p}=0, \tag{C3}$$

$$\frac{\partial\tilde{p}}{\partial\xi}+i\tilde{q}=0, \tag{C4}$$

and combining these two equations we get Eq. (4.22). Following a completely analogous reasoning to that of Appendix B we obtain the initial conditions (4.23) and (4.24).

APPENDIX D: DERIVATION OF EQ. (5.6)

We see from Eq. (5.3) that $\xi=0$ is a regular singular point of the differential equation. This allows us to seek a series solution of the form [23]

$$\tilde{p}(\omega,\xi,\tau)=\sum_{n=0}^{\infty}a_n(\omega,\tau)\xi^n. \qquad \text{(D1)}$$

The substitution of Eq. (D1) into Eq. (5.3) yields

$$\tilde{p}(\omega,\xi,\tau)=A(\omega,\tau)u(\xi)+B(\omega,\tau)v(\xi), \qquad \text{(D2)}$$

where A and B are to be determined from initial conditions, and

$$u(\xi)=1-\tfrac{1}{8}\sigma^2\xi^4+O(\xi^5), \qquad \text{(D3)}$$

$$v(\xi)=\xi^2+\tfrac{2}{3}(1+r)\xi^3+\tfrac{1}{4}(r^2+3r+2)\xi^4+O(\xi^5). \qquad \text{(D4)}$$

From Eq. (5.3) and Eqs. (D3) and (D4) we see that the Wronskian of $u(\xi)$ and $v(\xi)$ is

$$W\left[u(\xi),v(\xi)\right]=\frac{2\xi}{(1-\xi)^{1+r}}. \qquad \text{(D5)}$$

Combining Eqs. (5.4) and (5.5) with Eqs. (D2)–(D5) we obtain

$$\tilde{p}(\omega,\xi,\tau)=\frac{(1-\bar{\xi})^{1+r}}{2\bar{\xi}}\left[v'(\bar{\xi})u(\xi)-v(\xi)u'(\bar{\xi})\right] \times\exp\{-i\omega(x_0+y_0\bar{\xi}/\beta\}, \qquad \text{(D6)}$$

where $\bar{\xi}=1-e^{-\tau}$. In the original variables μ and t, Eq. (D6) yields Eq. (5.6).

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